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FREE AND FORCED VIBRATION OF A CANTILEVER BEAM CONTACTING WITH A RIGID CYLINDRICAL FOUNDATION

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In this paper, the free and forced vibration of a fixed-free Euler–Bernoulli beam in contact with a rigid cylindrical foundation is studied. One end of the beam is clamped at the top of the rigid cylindrical foundation and the other end is free. The vibrations are separated into upward and downward configurations, since a unilateral constraint is added by the cylindrical foundation. The partial differential equation, describing the transverse vibration of the cantilever beam, and the transversality condition, describing the contact position between the beam and the cylindrical foundation, are derived by calculus of variation and Hamilton's principle. This is a moving boundary problem since the unknown contact position has to be determined as part of the solutions. A Galerkin approximation is used to reduce the partial differential equation to a set of non-linear ordinary differential equations. The transient amplitudes and the phase planes of the vibration and the contact length are simulated by a Runge–Kutta algorithm. The effects of initial condition, radius parameter of the cylindrical foundation, externally static and harmonic excitations are investigated and discussed.

1. INTRODUCTION

The contact problems of an elastic layer (a thick plate or a beam) lying on an elastic or rigid subspace foundation have received a good deal of attention for a long time. Civelik and Erdogan [1, 2] solved the problem of an infinite strip resting on a rigid foundation and subjected to upward and downward directed loads. In their work, vertical load was used for solving a simple problem of lifting an elastic layer lying on a horizontal, rigid, frictionless subspace. Oden and Kikuchi [3] developed a finite element formulation, which is based on variational inequalities, for several classes of free boundary–value problems in mechanics. In particular, they considered contact problems in elasticity involving contact of one elastic body on the other. The criterion was derived for the selection of shape functions and the *a priori* error bounds were also established.

The problem involving beams of infinite length moving over supports, or acted upon by moving loads was presented in a series of papers by Adams [4–6], Adams and Bodgy [7] and Adams and Manor [8]. Adams [4, 5] studied an infinite elastic strip which rested on a flat rigid foundation and was loaded by upward and downward concentrated forces. Adams and Manor [8] investigated an infinite elastic beam which moved at a constant speed across a frictionless rigid step. Steady state solutions were obtained in closed form using both Euler–Bernoulli and Timoshenko beam models. The contact of an elastic cantilever beam over a flat rigid foundation was considered by Kikuchi and Oden [9]. The computed shape of deflection was compared with those furnished by the

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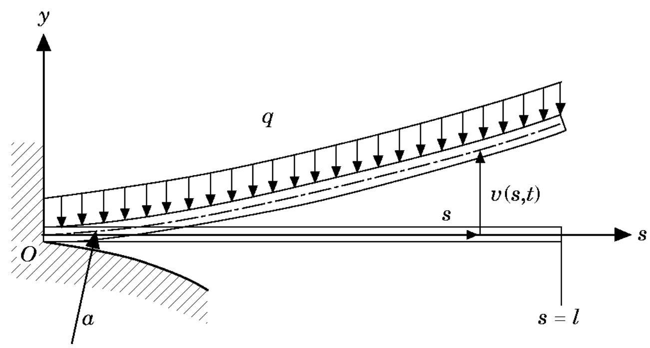

Figure 1. Physical model of the cantilever beam in the upward configuration with the rigid cylindrical foundation.

elementary beam theory. However, these studies were concerned with flat rigid foundation only.

Adams [10] studied an infinite elastic strip which moved at a constant speed across a frictionless rigid foundation possessing a step discontinuity. Adams and Manor [11] investigated an infinite beam which moved at a constant speed across a frictionless rigid base containing a cut-out. In these studies, they focused on the steady state solution of foundation discontinuity related to parameters of a beam system. Buffinton and Kane [12] studied the behavior of a uniform beam moving longitudinally at a prescribed rate over two bilateral supports. Equations of motions were formulated by regarding the supports as kinematical constraints. Recently, Tan *et al*. [13, 14] analyzed the vibration of a translating string, controlled through hydrodynamic bearing forces, by using the transfer function method. Interactions between the string response and the bearing film were described by the bearing impedance function. Wang and Kim [15] developed a new analysis method to solve the contact problem of a thin beam impacting against a stop. The method includes the full dynamics of all the elements of the system in the formulation.

In this paper, the results of the static equilibrium of a cantilever beam under a concentrated load of Fung and Fann [16] are extended to study the free and forced vibration of a cantilever beam in contact with a rigid cylindrical foundation. The physical model is shown in Figures 1 and 2 for the beam vibrating upward and downward respectively. When the beam vibrates upward, it involves the usual vibration problem of a cantilever beam. When the beam vibrates downward, a moving boundary exists due to the beam coming into contact with the cylindrical foundation. The geometric constraint of the cylindrical foundation, perpendicular and continuous conditions are introduced into

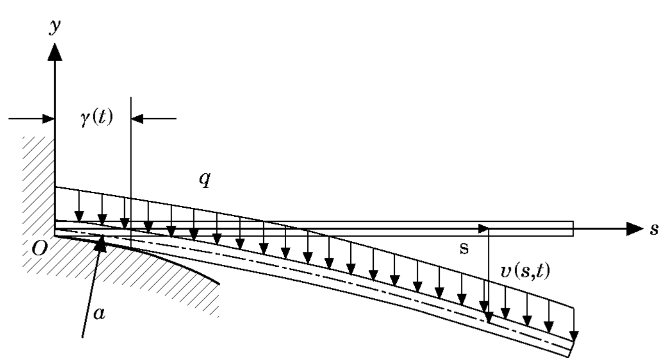

Figure 2. Physical model of the cantilever beam in the downward configuration with the rigid cylindrical foundation.

Hamilton's principle and calculus of variation to derive the governing equation and the moving boundary condition. The procedure for solving such equations involves a Galerkin approximation which reduces the partial differential equations to a set of non-linear ordinary differential equations.

2. FORMULATIONS OF PHYSICAL MODEL

In this paper the vibration of a cantilever beam clamped at the top of a rigid cylindrical foundation is studied. The beam is subjected to a general distributed load $q(s, t)$ and modeled by the Euler–Bernoulli theory. This problem is different from that of a simple cantilever beam. Since the vibrations of the beam in the upward and downward directions are quite different, they have to be analyzed separately. Primarily, configurations of this beam are divided into two parts: upward and downward vibration configurations. In the upward vibration configuration, as shown in Figure 1, the beam vibrates without the geometric constraint of the rigid cylindrical foundation. In the downward configuration, as shown in Figure 2, a part of the beam is constrained by its contact with the rigid cylindrical foundation, and the range of contact is time-dependent.

In the following sections, the governing equations and boundary conditions of the upward and downward configurations are formulated separately by using Hamilton's principle and calculus of variation. The notation appears in Appendix B.

2.1. UPWARD VIBRATION CONFIGURATION

The upward vibration configuration is shown in Figure 1. The Osy is the fixed co-ordinate with **i** and **j** unit vectors. When the beam vibrates upward, it involves the usual vibration problem of a cantilever beam and the formulation is easy. Assume the beam has length l and is made of homogeneous and uniform material. Vertical displacement $v(s, t)$ of the beam at location s is positive in the positive **j** direction.

Hamilton's principle is

$$0 = \int_{t_1}^{t_2}\left(\int_0^l \delta L(s, t; v_{,t}, v_{,ss})\,\mathrm{d}s + \int_0^l \delta W\,\mathrm{d}s\right)\mathrm{d}t, \tag{1}$$

where the Lagrangian density is

$$L(s, t; v_{,t}, v_{,ss}) = \tfrac{1}{2}\rho A v_{,t}^2(s, t) - \tfrac{1}{2}EIv_{,ss}^2(s, t),$$

and ρ is the mass density per unit length, A is the cross-sectional area of the beam, E is Young's modulus of the beam, I is the moment of inertia. $\delta W = -q\delta v$ is the virtual work done by the external force $q(s, t)$.

Since $\delta v(s, t)$ is arbitrary in the internal $0 < s < l$, and the boundary conditions are clamped at $s = 0$ and free at $s = l$, one obtains the governing equation

$$\rho A v_{,tt}(s, t) + EIv_{,ssss}(s, t) = -q, \qquad 0 < s < l, \tag{2}$$

with boundary conditions

$$v(0, t) = 0, \qquad v_{,s}(0, t) = 0, \qquad v_{,ss}(l, t) = 0, \qquad v_{,sss}(l, t) = 0. \tag{3a–d}$$

For convenience in studying the effects of system parameters, the following non-dimensional variables and parameters are defined:

$$V_1 = v/l, \qquad \xi = s/l, \qquad \tau = \omega_T t, \qquad \omega_T^2 = \pi^4 EI/\rho A l^4, \qquad \Omega_R = \omega_f/\omega_T$$
$$Q_0 = q_0 l^3/EI, \qquad Q_1 = q_1 l^3/EI, \qquad Q = ql^3/EI = Q_0 + Q_1\cos\Omega_R\tau, \tag{4}$$

where V_1 is used to describe the amplitude of the upward vibration and the external excitation $q(s,t) = q_0(s) + q_1 \cos \omega_f t$ is assumed. Then equation (2) becomes

$$V_{1,\tau\tau}(\xi,\tau) + (1/\pi^4)V_{1,\xi\xi\xi\xi}(\xi,\tau) = -Q, \qquad 0 < \xi < 1 \tag{5}$$

$$V_1(0,\tau) = 0, \qquad V_{1,\xi}(0,\tau) = 0, \qquad V_{1,\xi\xi}(1,\tau) = 0, \qquad V_{1,\xi\xi\xi}(1,\tau) = 0. \tag{6a–d}$$

By Galerkin's method, a suitable set of orthogonal basis functions $\phi_n(\xi)$, which satisfies geometric boundary conditions, is chosen to express the unknown displacement $V_1(\xi,\tau)$. Thus, one seeks a solution $V_1(\xi,\tau)$ of equation (5) in the form

$$V_1(\xi,\tau) = \sum_{n=1}^{\infty} f_n(\tau)\phi_n(\xi), \tag{7}$$

where $f_n(\tau)$ are the transient amplitudes. A typical choice for $\phi_n(\xi,\tau)$ in vibration problems are the normal modes of the associated linear problem. Thus one has

$$\phi_n(\xi) = \sqrt{\lambda_n/\Omega_n}\{[\cosh \lambda_n\xi - \cos \lambda_n\xi] + \Lambda_n[\sinh \lambda_n\xi - \sin \lambda_n\xi]\}, \tag{8}$$

in which λ_n is the natural frequency of the nth mode, and

$$\Lambda_n = (\sin \lambda_n - \sinh \lambda_n)/(\cos \lambda_n + \cosh \lambda_n),$$

$$\begin{aligned}\Omega_n = {} & \lambda_n + \tfrac{1}{4}(\cosh 2\lambda_n + \cos 2\lambda_n - 2) - \cos \lambda_n \sinh \lambda_n - \sin \lambda_n \cosh \lambda_n \\ & + \Lambda_n\{\tfrac{1}{2}(\cosh 2\lambda_n - \cos 2\lambda_n) - 2 \sin \lambda_n \sinh \lambda_n\} \\ & + \Lambda_n^2\{\tfrac{1}{4}(\sinh 2\lambda_n - \sin 2\lambda_n) + \cos \lambda_n \sinh \lambda_n - \sin \lambda_n \cosh \lambda_n\}.\end{aligned}$$

It can be verified that $\{\phi_n(\xi), n = 1, 2, \ldots\}$ forms an orthonormal basis. Substituting expression (7) into the equation of motion (5), taking the inner product with $\phi_m(\xi)$, $m = 1, 2, 3, \ldots$, and making use of boundary conditions (6) and the orthogonality properties of $\{\phi_n\}$, one obtains a system of a countable infinite number of linear ordinary differential equations for $f_m(\tau)$, $m = 1, 2, \ldots$:

$$\ddot{f}_m(\tau) + (\lambda_m^4/\pi^4)f_m(\tau) = -QK_{4m}, \tag{9}$$

where

$$K_{4m} = \int_0^1 \phi_m(\xi)\,\mathrm{d}\xi = \frac{1}{\lambda_m}\{\sinh \lambda_m - \sin \lambda_m + \Lambda_m(\cosh \lambda_n + \cos \lambda_n - 2)\}.$$

2.2. DOWNWARD VIBRATION CONFIGURATION

The downward vibration configuration is shown in Figure 2. When the beam vibrates downward, it will come into contact with the cylindrical foundation. At any time t, it is assumed the contact length is $\gamma(t)$ which is measured from $s = 0$. For small vibrations, it is reasonable to assume that the contact length is continuous, i.e., a multiple contact point problem is excluded. This is valid for lower modes of vibration.

For the convenience of analysis, one calls $s = \gamma(t)$ the separation point, γ^- and γ^+ are the positions just before and just after the separation point respectively. In the downward vibration configuration, the length region, $0 \leqslant s \leqslant \gamma^-$, comes into contact with the cylindrical foundation, and the length region, $\gamma^+ \leqslant s < l$ is free to vibrate without constraint. Before formulating the governing equation, the following characteristics of the geometric constraint, perpendicular and continuous conditions, and relationship of virtual variable must be determined first.

2.2.1. *Geometric constraint of displacement*

The geometric constraint will be investigated in the length region $0 \leqslant s \leqslant \gamma^-$. In Figure 2, Osy is the fixed frame which has origin O at the top of the cylindrical foundation. The position vector of any beam point s before deformation is

$$\mathbf{R}^{\mathbf{i}}(s) = s\,\mathbf{i} \tag{10}$$

and the displacement field is

$$\mathbf{U}(s, y, t) = -yv_{,s}(s, t)\,\mathbf{i} + v(s, t)\,\mathbf{j}, \tag{11}$$

where $\mathbf{i}$, $\mathbf{j}$ are unit vectors of the fixed co-ordinate, and $v(s, t)$ has a positive value in the positive $\mathbf{j}$ direction. This displacement in the $\mathbf{i}$ direction comes from the foreshortening effect. Therefore, the position vector after deformation is

$$\mathbf{R}^{\mathbf{f}}(s, y, t) = \mathbf{R}^{\mathbf{i}}(s) + \mathbf{U}(s, y, t) = (s - yv_{,s}(s, t))\,\mathbf{i} + v(s, t)\,\mathbf{j}. \tag{12}$$

The geometric constraint condition is defined by the fact that the radius length a is constant from the center of the cylindrical foundation to the neutral axis of the beam which is in contact with the cylindrical foundation in the length region $0 \leqslant s \leqslant \gamma^-$. Thus, the geometric constraint condition can be written as

$$C(v(s, t), s) = s^2 + v^2(s, t) + 2av(s, t) = 0, \qquad 0 \leqslant s \leqslant \gamma^-. \tag{13}$$

At point s of the neutral axis of the beam which is in contact with the foundation, the displacement $v(s, t)$ can be found from equation (13) as

$$v(s, t) = -a + \sqrt{a^2 - s^2}, \qquad 0 \leqslant s \leqslant \gamma^-. \tag{14}$$

2.2.2. *Perpendicular condition*

The radial vector from the center of the cylindrical foundation, $(0, -a)$, to the point s of the neutral axis of beam after deformation, $(s, v(s, t))$, is

$$\mathbf{R}_{\mathbf{r}} = s\mathbf{i} + (a + v(s, t))\,\mathbf{j}, \qquad 0 \leqslant s \leqslant \gamma^-. \tag{15}$$

From the geometric relationship that the radial vector $\mathbf{R}_{\mathbf{r}}$ is perpendicular to the tangential vector $\partial\mathbf{R}_{\mathbf{r}}/\partial s$, one obtains

$$\mathbf{R}_{\mathbf{r}} \cdot \partial\mathbf{R}_{\mathbf{r}}/\partial s = 0, \qquad 0 \leqslant s \leqslant \gamma^-. \tag{16}$$

From the above equation, one has

$$v_{,s}(s, t) = -s/(a + v(s, t)), \qquad 0 \leqslant s \leqslant \gamma^-. \tag{17}$$

which may be called the perpendicular condition. It is seen that the position s, the displacement v and the slope $v_{,s}$ are related in (17). It should be noted that $s = \gamma^-$ is the left-side point of the separation point and satisfies the relationship of (17).

2.2.3. *Continuous condition*

In the downward vibration, the beam touches the cylindrical foundation, and there exists a contact force in the interval $0 < s < \gamma^-(t)$. The beam in the other interval $\gamma^+(t) < s < l$ is free to vibrate. Thus the force exerted on the beam at $s = \gamma(t)$ is not continuous. It can be observed that the displacement $v(s, t)$ and slope $v_{,s}(s, t)$ of a beam under piecewisely distributed load or concentrated load are continuous everywhere (Gere and Timoshenko

[17]). This means that, the left- and right-hand of separation point $s = \gamma$ will have the same displacement and slope, i.e.,

$$v(\gamma^-, t) = v(\gamma^+, t) = -a + \sqrt{a^2 - \gamma^2}, \qquad v_{,s}(\gamma^-, t) = v_{,s}(\gamma^+, t) = -\gamma/(a + v(\gamma, t)). \tag{18a, b}$$

However, the term $v_{,ss}(s, t)$ may not be continuous at the point $s = \gamma(t)$ where the load is not continuous. Equations (18a, b) are the continuous conditions of displacement and slope respectively.

2.2.4. *Relationship of virtual variables*

In the following section, Hamilton's principle will be used to derive the dynamic equilibrium equation. Since both γ and $v(\gamma)$ are unspecified, the relationship of $\delta v(\gamma)$ and $\delta v_{,s}(\gamma)$ with respect to $\delta\gamma$ will be determined first. Although the separation point $s = \gamma$ and the displacement of the separation point $v(\gamma, t)$ are unspecified, they are related by the geometric constraint of (13). Since the position $s = \gamma$, the displacement $v(\gamma, t)$ and slope $v_{,s}(\gamma, t)$ are continuous, one has the relationship of virtual variables as

$$\delta\gamma^- = \delta\gamma^+, \qquad \delta v(\gamma^-, t) = \delta v(\gamma^+, t), \qquad \delta v_{,s}(\gamma^-, t) = \delta v_{,s}(\gamma^+, t). \tag{19}$$

Figure 3 shows the external and comparison curves in the v versus s plane. An extremal curve $v(s, t)$, terminating at point $s = \gamma$, and a neighboring comparison curve $v^*(s, t)$, terminating at the point $s = \gamma + \delta\gamma$, are shown. It is apparent that $\delta v(s, t) = v^*(s, t) - v(s, t)$ has meaning only in the interval $[\gamma + \delta\gamma, l]$, since $v^*(s, t)$ is not defined for $s \in (\gamma, \gamma + \delta\gamma)$. By inspection of Figure 3 one has

$$\delta\bar{v} = v^*(\gamma + \delta\gamma, t) - v(\gamma, t) \doteq v_{,s}(\gamma, t)\delta\gamma + \delta v(\gamma, t), \tag{20}$$

where the relation is equal to first order, i.e., the terms of the higher order than first order in δv and $\delta\gamma$ are neglected.

The point $s = \gamma + \delta\gamma$ of the comparison curve is also constrained to the geometric constrain condition (13), that is,

$$\begin{aligned} 0 &= C(v^*(\gamma + \delta\gamma, t), \gamma + \delta\gamma) = C(v(\gamma, t) + \delta\bar{v}, \gamma + \delta\gamma) \\ &= C(v(\gamma, t), \gamma) + (\partial C/\partial v(\gamma, t))\delta\bar{v} + (\partial C/\partial\gamma)\delta\gamma \\ &= 2[a + v(\gamma, t)]\delta\bar{v} + 2\gamma\delta\gamma. \end{aligned} \tag{21}$$

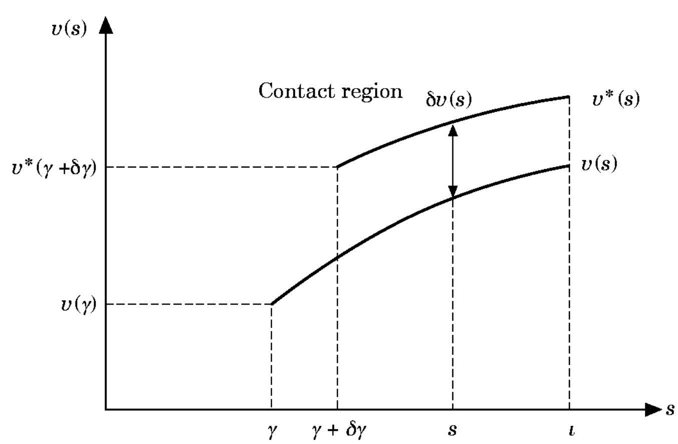

Figure 3. The extremal and comparison curves at the separation point.

Eliminating $\delta\bar{v}$ from (20) and (21), one gets

$$\delta v(\gamma, t) = -\{v_{,s}(\gamma, t) + \gamma/(a + v(\gamma, t))\}\delta\gamma, \tag{22}$$

which gives the relationship between $\delta v(\gamma, t)$ and $\delta\gamma$.

From the slope relationship (17), the slope constraint condition of the neutral axis of the beam which is in contact with the cylindrical foundation can be written as

$$0 = C'[s, v(s, t), v_{,s}(s, t)] = v_{,s} + s/(a + v(s, t)) \qquad 0 \leqslant s \leqslant \gamma^-. \tag{23}$$

The above equation could also be obtained directly by taking the derivative of (13) with respect to s.

$\delta\bar{v}_{,s}$ is defined as the slope difference between comparison and extremal curves, and written as

$$\delta\bar{v}_{,s}(\gamma, t) = v^*_{,s}(\gamma + \delta\gamma, t) - v_{,s}(\gamma, t) \doteq \delta v_{,s}(\gamma, t) + v_{,ss}(\gamma, t)\delta\gamma. \tag{24}$$

The slope at point $s = \gamma + \delta\gamma$ of the comparison curve is also constrained to the slope constraint (23), that is,

$$\begin{aligned} 0 &= C'[\gamma + \delta\gamma, v^*(\gamma + \delta\gamma, t), v^*_{,s}(\gamma + \delta\gamma, t)] = C'[\gamma + \delta\gamma, v(\gamma, t) + \delta\bar{v}, v_{,s}(\gamma, t) + \delta\bar{v}_{,s}] \\ &= C'[\gamma, v(\gamma, t), v_{,s}(\gamma, t)] + \frac{\partial C'}{\partial\gamma}\delta\gamma + \frac{\partial C'}{\partial v}\delta\bar{v} + \frac{\partial C'}{\delta v_{,s}}\delta\bar{v}_{,s} = \frac{1}{(a + v)}\delta\gamma - \frac{\gamma}{(a + v)^2}\delta\bar{v} + \delta\bar{v}_{,s}. \end{aligned} \tag{25}$$

It is seen that $\delta\gamma$, $\delta\bar{v}$ and $\delta\bar{v}_{,s}$ are related in (25). Using (20), (21) and (24), and eliminating $\delta\bar{v}_{,s}$ in (25), one obtains

$$\delta v_{,s}(\gamma^\pm, t) = -\left\{\frac{1 + v^2_{,s}(\gamma^\pm, t)}{a + v(\gamma^\pm, t)} + v_{,ss}(\gamma^\pm, t)\right\}\delta\gamma^\pm, \tag{26}$$

which states the relationship between $\delta v_{,s}(\gamma^\pm, t)$ and $\delta\gamma^\pm$.

The velocity at the point $s = \gamma(t)$ can be obtained by taking the total derivative of $v(s, t)$ with respect to t as

$$\mathrm{D}v(\gamma, t)/\mathrm{D}t = v_{,t}(\gamma(t), t) + v_{,s}(\gamma(t), t)\dot{\gamma} = \mathrm{D}/\mathrm{D}t\{-a + \sqrt{a^2 - \gamma^2(t)}\}. \tag{27}$$

Equation (27) can be rearranged as

$$v_{,t}(\gamma(t), t) = -\dot{\gamma}(t)\left[v_{,s}(\gamma(t), t) + \frac{\gamma(t)}{\sqrt{a^2 - \gamma^2(t)}}\right] = 0, \tag{28}$$

which means physically that the velocity of displacement in the y direction is zero at the separation point $s = \gamma(t)$.

2.2.5. *Hamiliton's Principle with moving boundary*

The formulation in the downward configuration is different from that in the upward configuration. The geometric constraint, the perpendicular and continuous conditions and the relationship of both $\delta v(\gamma)$ and $\delta v_{,s}(\gamma)$ with respect to $\delta\gamma$ will be used in the formulation. First, the total integral is divided into two domains including $0 \leqslant s \leqslant \gamma^-$ and $\gamma^+ \leqslant s \leqslant l$.

If one considers the geometric constraint (13) in the length $0 \leqslant s \leqslant \gamma^-$, Hamilton's principle becomes

$$0 = \delta \int_{t_1}^{t_2} \left[\int_0^{\gamma^-(t)} (L + \lambda C)\, \mathrm{d}s + \int_{\gamma^+(t)}^{l} L\, \mathrm{d}s \right] \mathrm{d}t$$

$$+ \int_{t_1}^{t_2} \left[\int_0^{\gamma^-(t)} + \int_{\gamma^+(t)}^{l} \right] \delta W\, \mathrm{d}s\, \mathrm{d}t, \tag{29}$$

where

$$L = \tfrac{1}{2}\rho A v_{,t}^2(s, t) - \tfrac{1}{2} EI v_{,ss}^2(s, t). \tag{30}$$

t_1 and t_2 are two arbitrary end times, and λ is the unknown function, which is referred to as Lagrangian multiplier in the length $0 \leqslant s \leqslant \gamma^-(t)$. The others are the same as those in the upward configuration.

Using the Lagrangian density (30) and the relationship of $\delta v(\gamma, t)$ and $\delta v_{,s}(\gamma, t)$ with respect to $\delta\gamma$, shown in equations (22) and (26), respectively, one obtains the governing equation

$$\rho A v_{,tt}(s, t) + EI v_{,ssss}(s, t) - 2\lambda[a + v(s, t)] = -q \qquad 0 < s < \gamma^- \tag{31a}$$

$$\rho A v_{,tt}(s, t) + EI v_{,ssss}(s, t) = -q \qquad \gamma^+ < s < l \tag{31b}$$

and the boundary conditions

$$v(0, t) = 0, \qquad v_{,s}(0, t) = 0, \qquad v(\gamma^{\pm}(t), t) = -a + \sqrt{a^2 - [\gamma^{\pm}(t)]^2}, \tag{32a–c}$$

$$v_{,s}(\gamma^{\pm}(t), t) = -\gamma^{\pm}(t)/(a + v(\gamma^{\pm}, t)), \qquad v_{,ss}(l, t) = 0, \qquad v_{,sss}(l, t) = 0. \tag{32d–f}$$

Since $\delta\gamma = \delta\gamma^- = \delta\gamma^+$, the corresponding transversality equation is also obtained as

$$\tfrac{1}{2}v_{,ss}^2(\gamma^-, t) - \tfrac{1}{2}v_{,ss}^2(\gamma^+, t) + \{(1 + v_{,s}^2(\gamma^-(t), t))/(a + v(\gamma^-(t), t))\} v_{,ss}(\gamma^-(t), t)$$
$$- \{(1 + v_{,s}^2(\gamma^+(t), t))/(a + v(\gamma^+(t), t))\} v_{,ss}(\gamma^+(t), t) = 0 \tag{33}$$

From equations (31–33), the following observations are made: (1) Equation (31a) describes the behavior of the beam in the contact domain $0 < s < \gamma^-$ with the boundary conditions (32a–d). The displacements of all points in this domain can be obtained from the geometric constraint (14). (2) Equation (31b) describes the beam vibration in the domain $\gamma^+ < s < l$ with the boundary conditions (32c–f). Equation (31b) is linear, boundary conditions (32c, d) are non-homogeneous while the transversality condition (33) is non-linear. (3) Equation (33) is called the transversality condition [18] which describes the condition that the separation point satisfies. Substituting $v(\gamma^{\pm}, t)$ and $v_{,s}(\gamma^{\pm}, t)$, shown in (32c, d), into (33), one will obtain $v_{,ss}(\gamma^{\pm}, t)$ which are functions of a and $\gamma(t)$ only.

The same non-dimensional variables and parameters as (4) of the upward vibration are used, except that $V_2 = v/l$ is used to describe the amplitude in the downward configuration. Non-dimensionalizing (31b), (32c–f) and (33), one obtains the following moving boundary problem:

$$V_{2,\tau\tau}(\xi, t) + (1/\pi^4) V_{2,\xi\xi\xi\xi}(\xi, t) = -Q, \qquad \Gamma^+ < \xi < 1, \tag{34}$$

$$V_{2,\xi\xi}^2(\Gamma^+, \tau) + \frac{2\alpha^2}{(\alpha^2 - \Gamma^2)^{3/2}} V_{2,\xi\xi}(\Gamma^+, \tau) + \frac{\alpha^4}{(\alpha^2 - \Gamma^2)^3} = 0, \tag{35}$$

$$V_2(\Gamma^+(\tau), \tau) = \Delta_1(\Gamma^+(\tau)), \qquad V_{2,\xi}(\Gamma^+(\tau), \tau) = \Delta_2(\Gamma^+(\tau)),$$
$$V_{2,\xi\xi}(1, \tau) = 0, \qquad V_{2,\xi\xi\xi}(1, \tau) = 0, \tag{36a–d}$$

where (35) is obtained from (33) and

$$\alpha = a/l, \qquad \Gamma = \gamma/l, \qquad \Delta_1(\Gamma^+(\tau)) = -\alpha + \sqrt{\alpha^2 - [\Gamma^+(\tau)]^2},$$

$$\Delta_2(\Gamma^+(\tau)) = -\Gamma^+(\tau)/(\alpha + V_2(\Gamma^+(\tau), \tau)).$$

3. APPROXIMATE SOLUTION OF THE DOWNWARD CONFIGURATION

3.1. VARIABLE TRANSFORMATION

Since the boundary conditions (36a) and (36b) are not homogeneous, it is difficult to apply directly Galerkin's method. Therefore, it is necessary to use variable transformation to eliminate the non-trivial boundary conditions.

Let the displacement function $V_2(\xi, \tau)$ be of the form

$$V_2(\xi, \tau) = V_3(\xi, \tau) + W(\xi, \tau), \tag{37}$$

where

$$W(\xi, \tau) = \Delta_2(\Gamma^+(\tau))(\xi - \Gamma^+) + \Delta_1(\Gamma^+(\tau)). \tag{38}$$

Substituting equations (37) and (38) into (34–36) and rearranging them, one obtains

$$V_{3,\tau\tau}(\xi, \tau) + (1/\pi^4)V_{3,\xi\xi\xi\xi}(\xi, \tau) = -W_{,\tau\tau}(\xi, \tau) - Q, \qquad \Gamma^+ < \xi < 1 \tag{39}$$

$$V^2_{3,\xi\xi}(\Gamma^+, \tau) + \frac{2\alpha^2}{(\alpha^2 - \Gamma^2)^{3/2}} V_{3,\xi\xi}(\Gamma^+, \tau) + \frac{\alpha^4}{(\alpha^2 - \Gamma^2)^3} = 0, \tag{40}$$

$$V_3(\Gamma^+, \tau) = 0, \qquad V_{3,\xi}(\Gamma^+, \tau) = 0, \qquad V_{3,\xi\xi}(1, \tau) = 0, \qquad V_{3,\xi\xi\xi}(1, \tau) = 0. \tag{41a–d}$$

After variable transformation (37), the non-homogeneous boundary conditions (36a, b) at $\xi = \Gamma^+$ become the clamped condition (41a, b). Equation (40) is the transversality condition and has two repeated solutions as:

$$V_{3,\xi\xi}(\Gamma^+, \tau) = -\alpha^2/(\alpha^2 - \Gamma^2)^{3/2} \tag{42}$$

It is seen from (42) that the transversality condition is dependent on the parameters α and Γ. The curvature in equation (42) is just equal to the curvature of the cylindrical foundation at $\xi = \Gamma^+$. In considering the special case, $\Gamma^+ = 0$, the curvature is equal to $-1/\alpha$.

3.2. APPROXIMATE SOLUTION

The approximation solution derived in this section is based on a Galerkin approximation with time-dependent basis function, which was used by Wang and Wei [19] to analyze vibration in a moving flexible robot arm, used by Yuh and Young [20] to study the dynamic modelling of an axial moving beam in rotation, and used by Fung and Cheng [21] to study the free vibration of a string/slider non-linear coupling system. To derive the approximate solution for the lateral vibration of a cantilever beam with the moving boundary at $\xi = \Gamma(\tau)$, first one considers the boundary-value problem

$$\psi_{n,\xi\xi\xi\xi}(\xi, \tau) - \beta_n^4(t)\psi_n(\xi, \tau) = 0, \qquad \Gamma^+ \leqslant \xi \leqslant 1 \tag{43}$$

$$\psi_n(\Gamma^+) = 0, \qquad \psi_{n,\xi}(\Gamma^+) = 0, \qquad \psi_{n,\xi\xi}(1) = 0, \qquad \psi_{n,\xi\xi\xi}(1) = 0. \tag{44a–d}$$

Since the spatial domain is time-dependent, both eigenfunction $\psi_n(\xi, t)$ and its corresponding eigenvalue $\beta_n(t)$ are time-dependent. An explicit expression $\psi_n(\xi, \tau)$ satisfying (43) is given by

$$\psi_n(\xi, \tau) = A_n[\cosh \beta_n(\xi - \Gamma) + \cos \beta_n(\xi - \Gamma)] + B_n[\cosh \beta_n(\xi - \Gamma) - \cos \beta_n(\xi - \Gamma)] + C_n[\sinh \beta_n(\xi - \Gamma) + \sin \beta_n(\xi - \Gamma)] + D_n[\sinh \beta_n(\xi - \Gamma) - \sin \beta_n(\xi - \Gamma)]. \tag{45}$$

where the constants A_n, B_n, C_n and D_n can be found from the four boundary conditions (44a–d). For a non-trivial solution of $\psi_n(\xi, \tau)$, one obtains the frequency equation

$$\cos \beta_n(1 - \Gamma^+) \cosh \beta_n(1 - \Gamma^+) = -1. \tag{46}$$

The first four roots of the above equation are

$$(1 - \Gamma^+)\beta_1 = \lambda_1 = 1{\cdot}875104, \qquad (1 - \Gamma^+)\beta_2 = \lambda_2 = 4{\cdot}694091,$$
$$(1 - \Gamma^+)\beta_3 = \lambda_3 = 7{\cdot}854757, \qquad (1 - \Gamma^+)\beta_4 = \lambda_4 = 10{\cdot}995541.$$

which can be written as the general form

$$\beta_n = \lambda_n/(1 - \Gamma^+), \qquad n = 1, 2, 3, 4, \ldots . \tag{47}$$

Note that λ_n is independent of τ.

Using boundary conditions (44) and the resulting shape functions are obtained

$$\psi_n(\xi, \tau) = \sqrt{\frac{\lambda_n}{[1 - \Gamma^+(\tau)]\Omega_n}}\left\{\left[\cosh \frac{\lambda_n}{(1 - \Gamma^+)}(\xi - \Gamma^+) - \cos \frac{\lambda_n}{(1 - \Gamma^+)}(\xi - \Gamma^+)\right] + \Lambda_n\left[\sinh \frac{\lambda_n}{(1 - \Gamma^+)}(\xi - \Gamma^+) - \sin \frac{\lambda_n}{(1 - \Gamma^+)}(\xi - \Gamma^+)\right]\right\} \tag{48}$$

where Ω_n is the same as in the upward vibration. Substituting $\Gamma^+ = 0$ into (48), the shape function is the same as $\phi_n(\xi)$ in equation (8) of the upward vibration. Thus, it is believed that the derivative is correct.

For applying Galerkin's method, the set of orthogonal basis function (48) is chosen to expand the unknown displacement $V_3(\xi, \tau)$. It can be verified that $\{\psi_n(\xi, \tau), n = 1, 2, \ldots\}$ forms a set of orthonormal basis functions. Thus, one seeks the solution $V_3(\xi, \tau)$ of equation (39) in the form

$$V_3(\xi, \tau) = \sum_{n=1}^{\infty} g_n(\tau)\psi_n(\xi, \tau) \tag{49}$$

Since $\psi_n(\xi, \tau)$ in (48) includes the moving boundary position $s = \gamma(t)$, it is the function of position ξ and time τ. On substituting (49) into the equation of motion (39) and taking the inner product from Γ^+ to 1 with $\psi_m(\xi, \tau)$, and using the boundary conditions (41) and the orthogonality properties of $\{\psi_n\}$, one obtains a system of linear time-varying ordinary differential equations for $g_m(\tau)$:

$$\ddot{g}_m(\tau) + \sum_{n=1}^{\infty} 2A_{nm}(\Gamma, \dot{\Gamma})\dot{g}_m(\tau) + \left\{\sum_{n=1}^{\infty} B_{nm}(\Gamma, \dot{\Gamma}, \ddot{\Gamma}) + \frac{\lambda_m^4}{\pi^4(1 - \Gamma^+)^4}\right\}g_m(\tau) = -I_m(\Gamma, \dot{\Gamma}, \ddot{\Gamma}) - Q(1 - \Gamma)K_{4m}, \qquad m = 1, 2, \ldots, \tag{50}$$

where the time-dependent coefficients are

$$A_{nm}(\Gamma, \dot{\Gamma}) = \int_{\Gamma^+}^{1} \dot{\psi}_n(\xi, \tau)\psi_m(\xi, \tau)\, \mathrm{d}\xi = J_{1nm}(\Gamma)\dot{\Gamma}, \tag{51a}$$

$$B_{nm}(\Gamma, \dot{\Gamma}, \ddot{\Gamma}) = \int_{\Gamma^+}^{1} \ddot{\psi}_n(\xi, \tau)\psi_m(\xi, \tau)\, \mathrm{d}\xi = J_{1nm}(\Gamma)\ddot{\Gamma} + J_{2nm}(\Gamma)\dot{\Gamma}^2, \tag{51b}$$

$$I_m(\Gamma, \dot{\Gamma}, \ddot{\Gamma}) = \int_{\Gamma^+}^{1} W_{,\tau\tau}(\xi, \tau)\psi_m(\xi, \tau)\, \mathrm{d}\xi = J_{3m}(\Gamma)\dot{\Gamma}^2 + J_{4m}(\Gamma)\ddot{\Gamma}, \tag{51c}$$

in which $J_{1nm}(\Gamma)$, $J_{2nm}(\Gamma)$, $J_{3m}(\Gamma)$ and $J_{4m}(\Gamma)$ are functions of Γ only and the detailed expressions are shown in Appendix A.

Substituting (49) into (42), one obtains the transversality condition:

$$\sum_{m=1}^{\infty} g_m(\tau)\psi_{m,\xi\xi}(\Gamma, \tau) = -\frac{\alpha^2}{(\alpha^2 - \Gamma^2)^{3/2}}. \tag{52}$$

The transient amplitudes of the upward and downward configurations will be obtained by the Runge–Kutta numerical integration method. It is easy to obtain the transient amplitude $f_m(\tau)$ of the upward vibration by integrating equation (9). However, the transient amplitude $g_m(\tau)$ of the downward vibration couples with $\Gamma(\tau)$, $\dot{\Gamma}(\tau)$ and $\ddot{\Gamma}(\tau)$ in (50). Also, the transversality condition (52) shows the coupling between $g_m(\tau)$ and $\Gamma(\tau)$.

4. NUMERICAL TECHNIQUE

It is obvious that $g_m(\tau)$ and $\Gamma(\tau)$ describe the vibration behavior of the downward configuration. The results could be solved from (50) and (52). Equation (50) is an ordinary differential equation for $\ddot{g}_m(\tau)$ and is coupled with $\Gamma(\tau)$, $\dot{\Gamma}(\tau)$ and $\ddot{\Gamma}(\tau)$. In order to solve equation (50), the equation for $\ddot{\Gamma}(\tau)$ should be obtained from (52) by differentiating twice with respect to τ first. Subsequently, the equations for $\ddot{g}_m(\tau)$ and $\ddot{\Gamma}(\tau)$ are combined to be solved simultaneously.

4.1. COMBINATION OF GOVERNING EQUATION AND TRANSVERSALITY CONDITION

Taking first and second time derivatives of the transversality condition (52), one obtains the following equations

$$\sum_{m=1}^{\infty} \eta_m g_m(\tau) = H_0(\Gamma), \qquad \sum_{m=1}^{\infty} \eta_m \dot{g}_m(\tau) = H_1(\Gamma)\dot{\Gamma},$$

$$\sum_{m=1}^{\infty} \eta_m \ddot{g}_m(\tau) = H_1(\Gamma)\Gamma + H_2(\Gamma)\dot{\Gamma}^2, \tag{53a–c}$$

where

$$\eta_m = -\frac{2\lambda_m^{5/2}}{\Omega_m^{1/2}\alpha^2}, \qquad H_0(\Gamma) = \frac{(1-\Gamma)^{5/2}}{(\alpha^2 - \Gamma^2)^{3/2}}, \qquad H_1(\Gamma) = \frac{(1-\Gamma)^{3/2}(6\Gamma - \Gamma^2 - 5\alpha^2)}{2(\alpha^2 - \Gamma^2)^{5/2}},$$

$$H_2(\Gamma) = \frac{(1-\Gamma)^{3/2}(3-\Gamma)}{(\alpha^2 - \Gamma^2)^{5/2}} + \frac{(1-\Gamma)^{1/2}(6\Gamma - \Gamma^2 - 5\alpha^2)}{(\alpha^2 - \Gamma^2)^{5/2}}\left[-\frac{3}{4} + \frac{5\Gamma(1-\Gamma)}{2(\alpha^2 - \Gamma^2)}\right].$$

In equation (53a), multiplying both sides by $(\alpha^2 - \Gamma^2)^{3/2}$ and squaring, a six-degree polynomial of $\Gamma(\tau)$ will be obtained. From physical examination, the value of $\Gamma(\tau)$ must be a positive real root. Equation (53a) shows the relationship between $g_m(\tau)$ and $\Gamma(\tau)$. Once $g_m(\tau)$ is known, $\Gamma(\tau)$ can be determined numerically. Substituting $\Gamma(\tau)$ and $\dot{g}(\tau)$ into equation (53b) will solve for the velocity $\dot{\Gamma}(\tau)$ of the separation point. From equation (53c), one obtains the acceleration relationship of the transversality condition as

$$\sum_{m=1}^{\infty} \eta_m \ddot{g}_m - H_1(\Gamma)\ddot{\Gamma} = H_2(\Gamma)\dot{\Gamma}^2. \tag{54}$$

Substituting (51a, c) into (50), one obtains the acceleration relationship of the governing equation as

$$\begin{aligned} \ddot{g}_m + \left\{ J_{4m}(\Gamma) + \sum_{n=1}^{\infty} J_{1nm}(\Gamma) g_m \right\} \ddot{\Gamma} \\ = \left\{ -\sum_{n=1}^{\infty} 2J_{1nm}(\Gamma)\dot{g}_m - \left[\sum_{n=1}^{\infty} J_{2nm}(\Gamma) + J_{3m}(\Gamma) \right] \dot{\Gamma} \right\} \dot{\Gamma} \\ -(\lambda_m^4/\pi^4(1-\Gamma)^4) g_m - Q(1-\Gamma)K_{4m}. \end{aligned} \tag{55}$$

It is seen from (54) and (55) that $\ddot{g}_m(\tau)$ and $\ddot{\Gamma}(\tau)$ are functions of $\dot{g}_m(\tau)$, $g_m(\tau)$, $\dot{\Gamma}(\tau)$ and $\Gamma(\tau)$. Thus, the solutions $g_m(\tau)$ and $\Gamma(\tau)$ could be obtained by integrating (54) and (55) simultaneously.

Combining equation (54) with (55) and writing in matrix form, one obtains

$$\begin{bmatrix} -H_1(\Gamma) & \eta_1 & \eta_2 & \cdots & \cdot & \eta_m \\ J_{41}(\Gamma) + \sum_{j=1}^{n} J_{1j1}(\Gamma)_{g_1} & 1 & 0 & \cdots & \cdot & 0 \\ J_{42}(\Gamma) + \sum_{j=1}^{n} J_{1j2}(\Gamma)_{g_2} & 0 & \cdot & \cdots & & \cdot \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \cdot & \cdot & & \cdots & \cdot & 0 \\ J_{4m}(\Gamma) + \sum_{j=1}^{n} J_{1jm}(\Gamma) g_m & 0 & \cdot & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\Gamma} \\ \ddot{g}_1 \\ \cdot \\ \vdots \\ \cdot \\ \ddot{g}_m \end{bmatrix}$$

$$= \begin{bmatrix} H_2(\Gamma)\dot{\Gamma} & 0 & \cdots & 0 \\ -\sum_{j=1}^{n} 2J_{1j1}\dot{g}_1 - \left(\sum_{j=1}^{n} J_{2j1}(\Gamma) + J_{31}(\Gamma) \right)\dot{\Gamma} & 0 & \cdots & 0 \\ -\sum_{j=1}^{n} 2J_{1j2}\dot{g}_2 - \left(\sum_{j=1}^{n} J_{2j2}(\Gamma) + J_{32}(\Gamma) \right)\Gamma & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_{j=1}^{n} 2J_{1jm}\dot{g}m - \left(\sum_{j=1}^{n} J_{2jm}(\Gamma) + J_{3m}(\Gamma) \right)\dot{\Gamma} & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} \dot{\Gamma} \\ \dot{g}_1 \\ \\ \\ \\ \end{matrix} \begin{bmatrix} \\ \\ \cdot \\ \vdots \\ \dot{g}_m \end{bmatrix}$$

$$
+\begin{bmatrix}
0 & \cdot & \cdot & \cdot & \cdots & 0 \\
\cdot & -\lambda_1^4/\pi^4(1-\Gamma)^4 & 0 & \cdot & \cdots & 0 \\
\cdot & 0 & -\lambda_2^4/\pi^4(1-\Gamma)^4 & \cdot & \cdot & \cdot \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdot & \cdot & \cdot & 0 & \cdot & \cdot \\
0 & 0 & \cdot & \cdot & 0 & -\lambda_m^4/\pi^4(1-\Gamma)^4
\end{bmatrix}
$$

$$
\begin{bmatrix} \Gamma \\ g_1 \\ \cdot \\ \vdots \\ \\ g_m \end{bmatrix} - Q(1-\Gamma) \begin{bmatrix} 0 \\ K_{41} \\ K_{42} \\ \vdots \\ \cdot \\ K_{4m} \end{bmatrix} \tag{56}
$$

With the suitable initial conditions $g_m(0)$ and $\Gamma(0)$, $g_m(\tau)$ and $\Gamma(\tau)$ in (56) could be solved simultaneously. This procedure, combining the accelerations of both the governing equation and the transversality condition is similar to that used by Haug [23] in the matrix formulation of the constrained equations of motion.

4.2. TRANSFER BETWEEN UPWARD AND DOWNWARD CONFIGURATIONS

In this paper, the upward and downward configurations are separately analyzed in the vibration analysis. The transversality condition (52) is used to judge whether the vibrating beam is in the upward or downward configuration. It is convenient to call it the transferring time; the time for transferring between upward and downward configurations. At the transferring time, the following conditions should be satisfied: (1) transferring from the upward configuration to the downward one, the contact length begins with $s = \Gamma = 0$ and increases with time when the beam vibrates downward continuously. Thus, the velocity $\dot{\Gamma}(\tau_t)$ of the contact position, where τ_t is the transferring time, should not be zero and could be determined from the transferring condition (3). (2) At the instant of transferring between upward and downward configurations, the displacements are equal, i.e., $V_1(\xi, \tau_t) = V_2(\xi, \tau_t)$. It can be written in the explicit form as

$$
\sum_{m=1}^{\infty} f_m(\tau_t)\phi_m(\xi) = \sum_{m=1}^{\infty} g_m(\tau_t)\psi_m(\xi, \tau_t) + W(\xi, \tau_t). \tag{57}
$$

Substituting $\Gamma = 0$ into (57) and using $\phi_m(\xi) = \psi_m(\xi, \tau_t)|_{\Gamma=0}$ and $W(\xi, \tau_t)|_{\Gamma=0} = 0$, one obtains

$$
f_m(\tau_t) = g_m(\tau_t). \tag{58}
$$

Equation (58) states that the amplitudes of the upward and downward configurations are equal at the transferring time τ_t. (3) The velocity of the vibrating beam in the upward configuration must transfer to that in the downward configuration. Thus, at the transferring time, it is necessary that $V_{1,\tau}(\xi, \tau_t) = V_{2,\tau}(\xi, \tau_t)$. Expanding in terms of eigenfunctions, we obtain

$$\sum_{m=1}^{\infty} \dot{f}_m(\tau_t)\phi_m(\xi) = \sum_{m=1}^{\infty} \{\dot{g}_m(\tau_t)\psi_m(\xi, \tau_t) + g_m(\tau_t)\dot{\psi}_m(\xi, \tau)\} + \dot{\Delta}_2(\tau_t)(\xi - \Gamma). \tag{59}$$

At the transferring time, Γ must be equal to zero. Thus, one has $\phi_m(\xi) = \psi_m(\xi, \tau_t)$. Taking the inner product of equation (59) with $\psi_n(\xi)$ from Γ to 1 and using the orthogonal properties of $\{\psi_n\}$, one obtains a system of linear time-varying ordinary differential equations:

$$\dot{f}_n(\tau_t) = \dot{g}_n(\tau_t) + \left\{\sum_{m=1}^{\infty} \frac{\Omega_n}{\lambda_n} J_{1nm}(0)g_m(\tau_t) - \frac{K_{5n}}{\alpha}\sqrt{\frac{\Omega_n}{\lambda_n}}\right\}\dot{\Gamma}(\tau_t), \tag{60}$$

which relates $\dot{f}_n(\tau_t)$, $\dot{g}_n(\tau_t)$, and $\dot{\Gamma}(\tau_t)$. It is seen that from condition (2) the amplitude $f_m(\tau_t)$ transfers to $g_m(\tau_t)$ with condition (1) $\Gamma(\tau_t) = 0$. From (60), the velocity $\dot{f}_m(\tau_t)$ transfers to $\dot{g}_m(\tau_t)$ and $\dot{\Gamma}(\tau_t)$. Because $\dot{\Gamma}(\tau_t) \neq 0$, an extra condition is needed to satisfy such a transformation in (60). It is clear that the time derivative of the transversality condition (53b) also holds. Combining equation (60) with (53b), one obtains the following matrix form:

$$\begin{bmatrix} -H_1(\Gamma) & \eta_1 & \eta_2 & \cdots & \cdot & \eta_n \\ \frac{\Omega_1}{\lambda_1}\sum_{j=1}^{m} J_{11j}(\Gamma)g_j(\tau_t) - \frac{K_{51}}{\alpha}\sqrt{\frac{\Omega_1}{\lambda_1}} & 1 & 0 & \cdots & \cdot & 0 \\ \frac{\Omega_2}{\lambda_2}\sum_{j=1}^{m} J_{12j}(\Gamma)g_j(\tau_t) - \frac{K_{52}}{\alpha}\sqrt{\frac{\Omega_2}{\lambda_2}} & 0 & 1 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \vdots & \vdots & & \cdots & \ddots & \\ \cdot & \cdot & \cdot & \cdots & \cdot & 0 \\ \frac{\Omega_n}{\lambda_n}\sum_{j=1}^{m} J_{1nj}(\Gamma)g_j(\tau_t) - \frac{K_{5n}}{\alpha}\sqrt{\frac{\Omega_n}{\lambda_n}} & 0 & \cdot & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\Gamma} \\ \dot{g}_1 \\ \dot{g}_2 \\ \cdot \\ \vdots \\ \cdot \\ \dot{g}_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdot & \cdot & \cdots & \cdot & 0 \\ \cdot & 1 & 0 & \cdots & \cdot & 0 \\ \cdot & 0 & 1 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & 0 \\ 0 & 0 & \cdot & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \dot{f}_1 \\ \dot{f}_2 \\ \cdot \\ \vdots \\ \cdot \\ \dot{f}_n \end{bmatrix}. \tag{61}$$

Equation (61) is used to calculate the values of $\dot{\Gamma}$, $\dot{g}_1$, $\dot{g}_2, \ldots, \dot{g}_n$ for the given values $\dot{f}_1, \dot{f}_2, \ldots, \dot{f}_n$. Details of J_{1nm} and K_{5n} are shown in the Appendix. It is worth noting that the n values of $\dot{f}_1, \dot{f}_2, \ldots, \dot{f}_n$ in equation (61) are used to determine the $n+1$ values

$\dot{\Gamma}, \dot{g}_1, \dot{g}_2, \ldots, \dot{g}_n$ at the transferring time. (4) When the beam begins to come into contact with the cylindrical foundation, one has $V_{1,\xi\xi}(0, \tau_t) = V_{2,\xi\xi}(0, \tau_t) = -1/\alpha$, i.e., the curvature at the point $\xi = \Gamma = 0$ is equal to $-1/\alpha$. Taking twice the derivative of both $V_1(\xi, \tau)$ and $V_2(\xi, \tau)$ with respect to ξ and setting $\xi = \Gamma = 0$, one obtains

$$\sum_{m=1}^{\infty} f_m(\tau_t) \frac{2\lambda_m^{5/2}}{\Omega_m^{1/2}} = \sum_{m=1}^{\infty} g_m(\tau_t) \frac{2\lambda_m^{5/2}}{\Omega_m^{1/2}} = -\frac{1}{\alpha}. \quad (62)$$

Equation (62) could be used to determine the transferring time τ_t in the numerical algorithm.

4.3. INITIAL CONDITIONS

If the initial conditions $f_m(0)$ and $\dot{f}_m(0)$ are given in the upward vibration configuration the motion is obtained completely by integrating equation (9) by the Runge–Kutta method. When vibration is in the upward configuration, the conditions $V_1(0, \tau) = V_{1,\xi}(0, \tau) = 0$ always hold at the clamped end $\xi = 0$.

If the initial conditions $f_m(0)$ and $\dot{f}_m(0)$ are so small that the beam does not come into contact with the cylindrical foundation, the vibration is the same as that of a cantilever beam without geometric constraint.

When the beam begins to touch the cylindrical foundation, from condition (4), the criterion $V_{3,\xi\xi}(0^+, \tau_t)$ is equal to $-1/\alpha$. For simplicity, taking $m = 1$ in (62), the criterion of contact condition (4) becomes

$$g_1(\tau_t) = -\Omega_1^{1/2}/2\lambda_1^{5/2}\alpha \quad (63)$$

The relationship between $g_1(\tau_t)$ and α is plotted in Figure 4. If the initial conditions $\dot{g}_1(0) = 0$ and $g_1(0) > g_1(\tau_t)$ are applied in the downward vibration, the beam will not come into contact with the cylindrical foundation. The contact and no contact regions are shown in Figure 4.

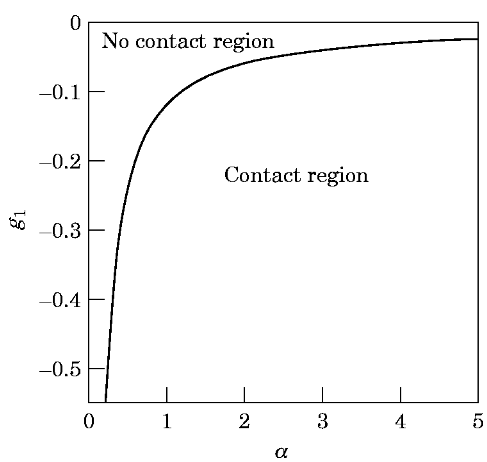

Figure 4. The contact and no contact regions shown in the plane of the transient amplitude g_1 versus the dimensionless cylindrical radius parameter α.

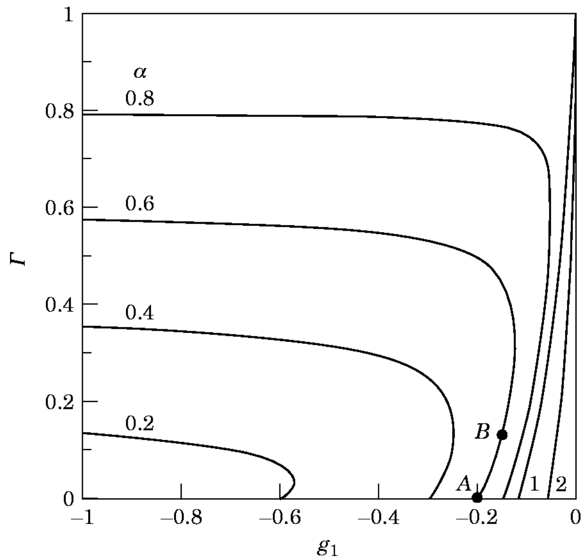

Figure 5. Relationship between the transient amplitude g_1 and the contact length Γ of the transversality condition with $\alpha = 0{\cdot}2$, $0{\cdot}4$, $0{\cdot}6$, $0{\cdot}8$, 1 and 2.

When the beam vibrates in the downward configuration, the transversality condition always holds at $\xi = \Gamma(\tau)$. Taking $m = 1$, one writes (52) in the form

$$g_1\lambda_1^{5/2}/\Omega_1^{1/2} = -\alpha^2(1-\Gamma)^{5/2}/2(\alpha^2-\Gamma^2)^{3/2}, \tag{64}$$

which describes the relationship of g_1, α and Γ. Figure 5 is plotted from (64) with various values of the radius parameter α of the cylindrical foundation.

In Figure 5, considering the case of free vibration with $\alpha = 0{\cdot}6$ and the initial conditions $f_1(0) > 0{\cdot}2$ and $\dot{f}_1(0) = 0$ given in the upward configuration, the beam vibrates downward and $f_1(\tau)$ decreases and assumes a negative value. When $f_1(\tau)$ reaches the value $-\Omega_1^{1/2}/2\lambda_1^{5/2}\alpha \approx 0{\cdot}2$, i.e., the curvature $V_{1,\xi\xi}(0^+, \tau) = -1/\alpha$, point A is the transferring point and the time $\tau = \tau_t$. It is time for the downward configuration to begin; $\dot{\Gamma}(\tau_t)$ and $\dot{g}_1(\tau_t)$ could be obtained from (61). With these initial conditions $g_1(\tau_t) = f_1(\tau_t)$, $\dot{g}_1(\tau_t)$, $\Gamma(\tau_t) = 0$ and $\dot{\Gamma}(\tau_t)$ and integrating (56), $g_1(\tau)$ and $\Gamma(\tau)$ can be obtained. The solution is along the path from A to B as shown in Figure 5, where point B is the assumed point with maximum contact length. Now, the kinetic energy of the beam in the upward configuration is transferred to an increase in the contact length and the vibration of the beam in the downward configuration. Thus, after the contact occurs, the $\Gamma(\tau)$ increases and $g_1(\tau)$ decreases along the path from A to B.

From point B, the beam vibrates along the path to point A, $g_1(\tau)$ increases to $-\Omega_1^{1/2}/2\lambda_1^{5/2}\alpha$ and $\Gamma(\tau)$ decreases to zero. Then, the beam returns to the upward vibration and completes one cycle motion. Based on the energy conservation of a free vibration beam, the results shown in Figure 5 are correct.

5. NUMERICAL RESULTS AND DISCUSSION

The external excitation $Q = Q_0 + Q_1 \cos \Omega_R\tau$ may be divided into several kinds of conditions: (a) $Q = 0$; free vibration: (b) $Q_1 = 0$, $Q = Q_0$; static forced vibration: (c) $Q_0 = 0$, $Q = Q_1 \cos \Omega_R\tau$; harmonic forced vibration: (d) Without kinetic energy $\frac{1}{2}\rho A v_{,t}^2$, i.e., neglecting $\rho A v_{tt}$ in (31a, b), the static equilibrium equations, boundary conditions and transversality condition are, respectively.

$$EIv_{,ssss}(s) - 2\lambda[a + v(s)] = -q, \qquad 0 < s < \gamma^-, \qquad EIv_{,ssss}(s) = -q, \qquad \gamma^+ < s < l, \tag{65a, b}$$

$$v(0) = 0, \qquad v_{,s}(0) = 0, \qquad v_{,ss}(l) = 0, \qquad v_{,sss}(l) = 0, \tag{66a–d}$$

$$v_{,ss}^2(\gamma^+)(v_{,ss}(\gamma^+) + a^2/(a^2 - \gamma^2)^{3/2}) = 0 \tag{67}$$

The static equilibrium of a cantilever beam on a rigid cylindrical foundation was studied by Chen [22]. It is seen from (42) that the transversality condition is dependent on the parameters α and Γ, but not on the external excitation. Thus, free and forced vibration and static equilibrium will have the same transversality condition (42).

5.1. FREE VIBRATION

For free vibration, the external excitation is absent, $Q = 0$. In Figure 6, the effect of the rigid cylindrical foundation on the transient amplitude of free vibration is investigated. Figure 6(a) shows the transient amplitude at the free end $\xi = 1$. Solid lines represent the transient amplitude with the cylindrical foundation, while dash-dot lines are used for those without the cylindrical foundation. It is seen that the transient amplitudes in the upward configuration are the same, but in the downward configuration the transverse displacement with the cylindrical foundation is larger than that without the cylindrical foundation. Also, the period with a cylindrical foundation is shorter than that without a cylindrical foundation. Since the damping effect is neglected in the formulation, the energy is conservative for the free vibration beam. The phase plane of amplitude at the free end, as shown in Figure 6(b), forms a closed cycle for the free vibration beam. The solid line represents the closed cycle with the cylindrical foundation while the dash-dot line is used for that without the cylindrical foundation. The transient transverse amplitude and its phase plane are shown in Figures 6(e) and 6(f) respectively, where f_1 and g_1 represent the transverse amplitudes in the upward and downward configurations. Since the kinetic energy of the beam in the upward configuration is transferred to an increase in the contact length and the vibration of the beam in the downward configuration, it follows that, after contact occurs, $\Gamma(\tau)$ increases (Figure 6(c)) and $g_1(\tau)$ decreases (Figure 6(e), solid line).

In Figures 7(a)–(f), one compares the transient amplitudes and the phase planes for various initial conditions, which are given at the free end as $V_1(1, 0) = 0{\cdot}6$, $0{\cdot}3$ and $0{\cdot}15$. From these figures, one finds that as the initial condition increases, the contact length increases and the period of transient amplitude decreases. There is no contact length for the initial condition $V_1(1, 0) = 0{\cdot}15$. It is seen from Figure 7(c) that the contact time will decrease as the initial condition decreases.

In Figures 8(a)–(f), the effect of the non-dimensionless radius parameter α of a cylindrical foundtion on the vibration amplitudes and phase planes is shown. It is found in Figures 8(a) and 8(b) that as the radius parameter of a cylindrical foundation decreases, the period of the beam vibration increases. With the same initial conditions, as the radius parameter of the cylindrical foundation increases, the contact length Γ increases, as shown in Figure 8(c) and 8(d). Figures 8(e) and 8(f) show the transient transverse amplitude and its phase plane, respectively.

5.2. FORCED VIBRATION

In Figures 9(a)–(f), is shown the transient amplitudes and phase planes with various static loads $Q_0 = 0{\cdot}01$, $0{\cdot}02$ and $0{\cdot}03$. From Figures 9(a)–(f), one finds that the increase in static load is related to an increase in period, transient amplitude (Figures 9(a), (b)) and contact length (Figures 9(c), (d)). Figures 9(e) and 9(f) show the transient amplitude and its phase plane respectively.

If the system is subjected to harmonic excitation and its excitation frequency is equal to its natural frequency, resonance will happen and the transient amplitude increases

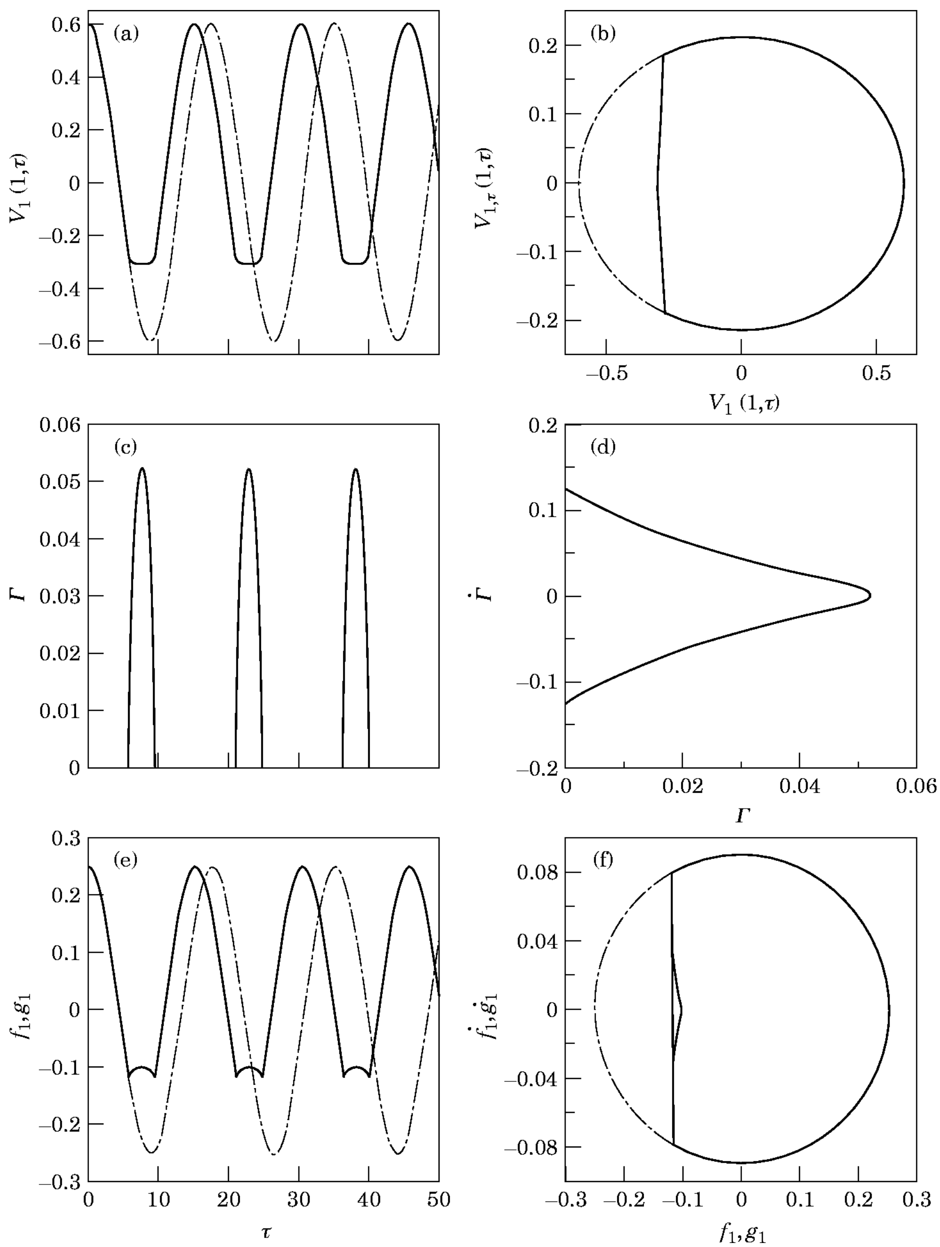

Figure 6. Free vibration of the cantilever beam with $\alpha = 1$, $V_1(1, 0) = 0{\cdot}6$, $V_{1,\tau}(1, 0) = 0$, $Q_0 = 0$, $Q_1 = 0$. (a) Transient amplitude at $\xi = 1$; (b) phase plane of the amplitude at $\xi = 1$; (c) transient contact length; (d) phase plane of the contact length; (e) transient amplitude; (f) phase plane of the amplitude. —, with the cylindrical foundation; – · –, without the cylindrical foundation.

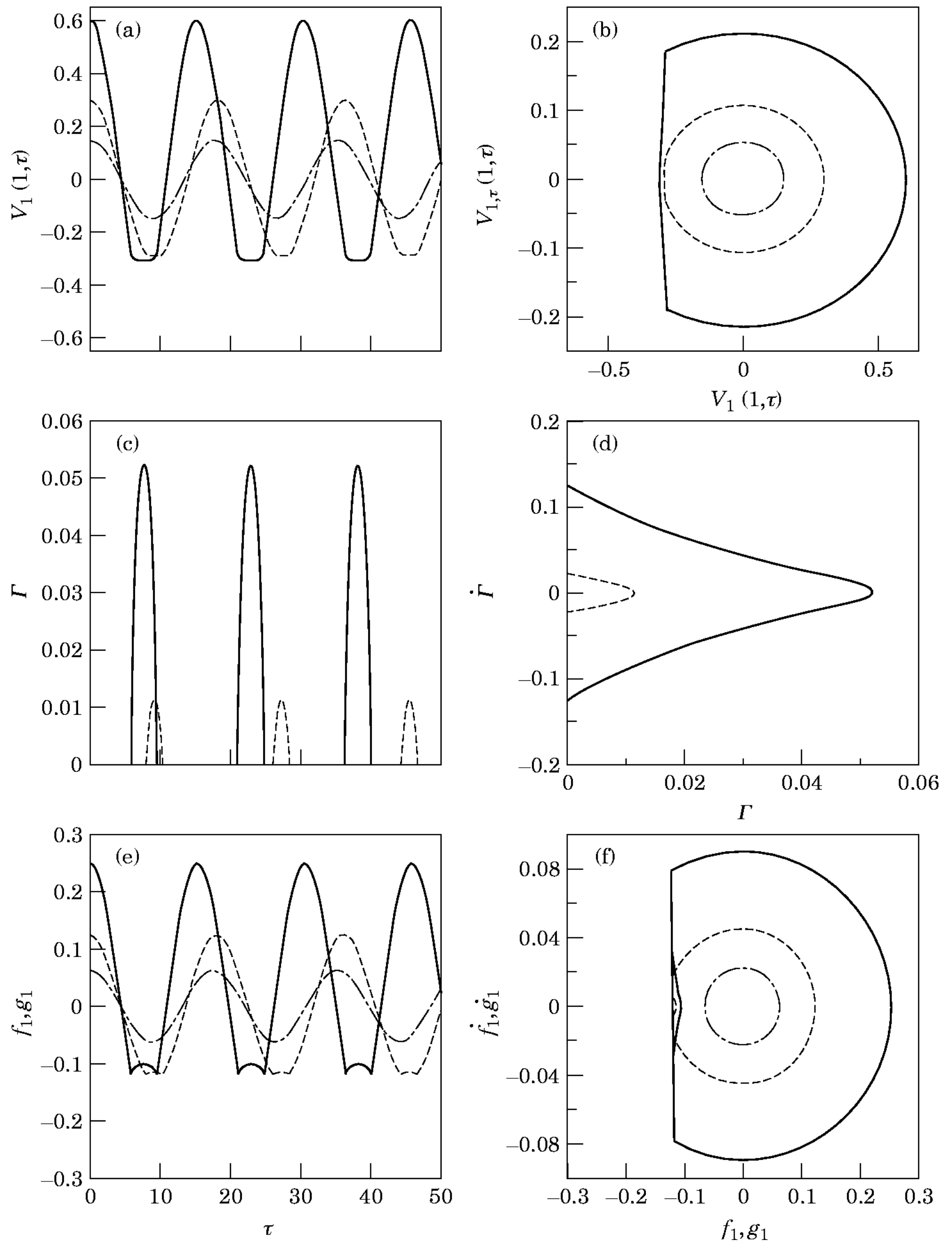

Figure 7. Transient amplitudes and phase planes for various initial conditions. (a) Transient amplitude at $\xi = 1$; (b) phase plane of the amplitude at $\xi = 1$; (c) transient contact length; (d) phase plane of the contact length; (e) transient amplitude; (f) phase plane of the amplitude. —, $V_1(1, 0) = 0{\cdot}6$; – –, $V_1(1, 0) = 0{\cdot}3$; – · –, $V_1(1, 0) = 0{\cdot}15$; the other parameters are as Figure 6a.

infinitely (not shown here). However, Figures 10(a)–(f) show the beating phenomenon of the transient amplitude and the contact length of the vibrating beam. In considering the cylindrical foundation the results for various excitation frequencies are shown. The transient amplitude and contact length are largest in Figures 10(a) and 10(b) for the resonant vibration. It is worth noting that the natural frequency changes when the beam comes into contact with the cylindrical foundation.

If the forcing frequency is close to, but not exactly equal to, the natural frequency of the system, a phenomenon known as beating may occur. In this kind of vibration, the amplitude builds up and then diminishes in a regular pattern. If all of the initial conditions are taken as zero, the transient amplitude of a vibrating beam is reduced to

$$f(\tau) = -(Q_1 K_{41}/(\Omega_N^2 - \Omega_R^2)) \sin \varepsilon\tau \sin \Omega_{R^\tau} \tag{68}$$

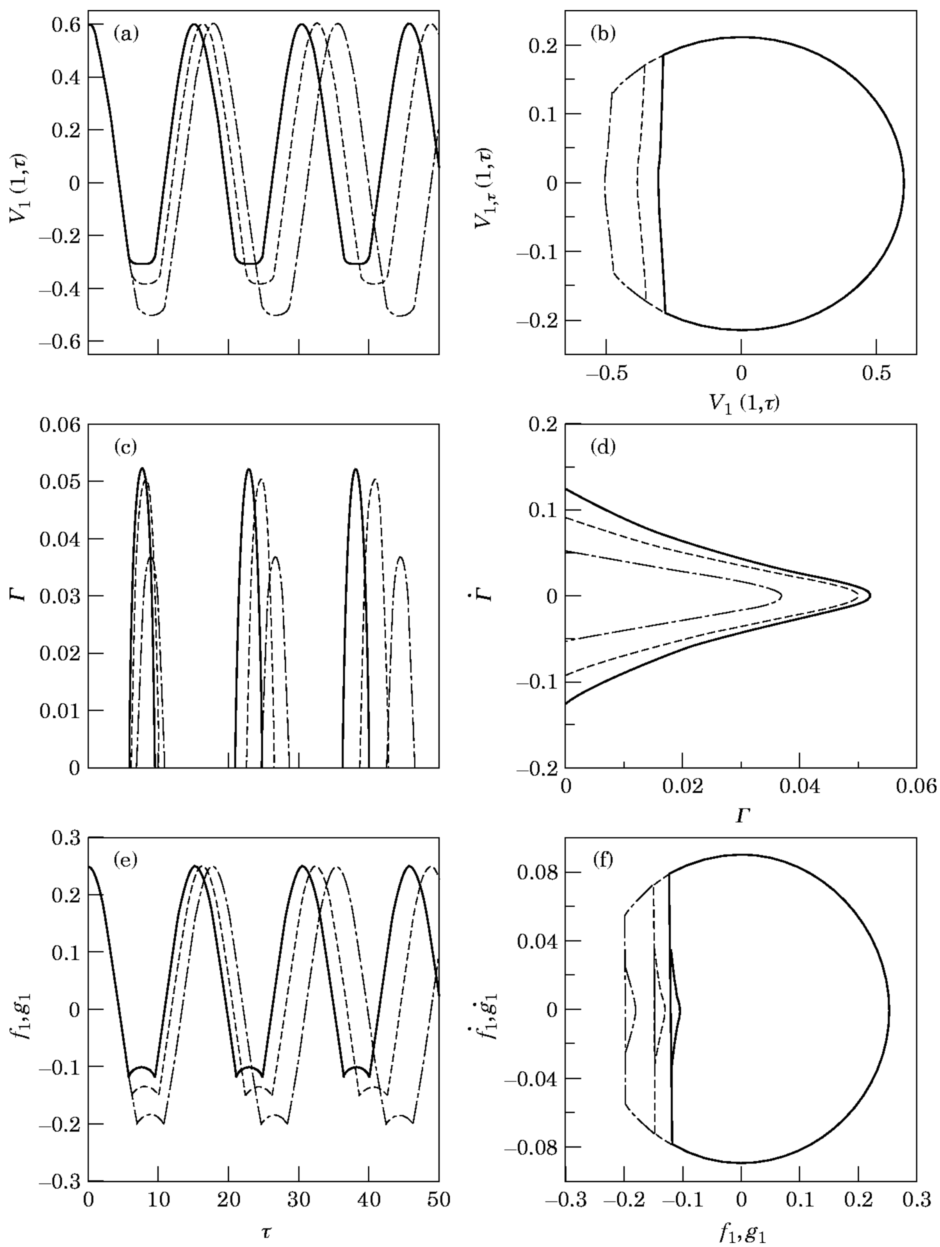

Figure 8. Transient amplitudes and phase planes for various radius parameters of the cylindrical foundation. (a) Transient amplitude at $\xi = 1$; (b) phase plane of the amplitude at $\xi = 1$; (c) transient contact length; (d) phase plane of the contact length; (e) transient amplitude; (f) phase plane of the amplitude. —, α: 1, – –, 0·8; – · –, 0·6; the other parameters are as Figure 6a.

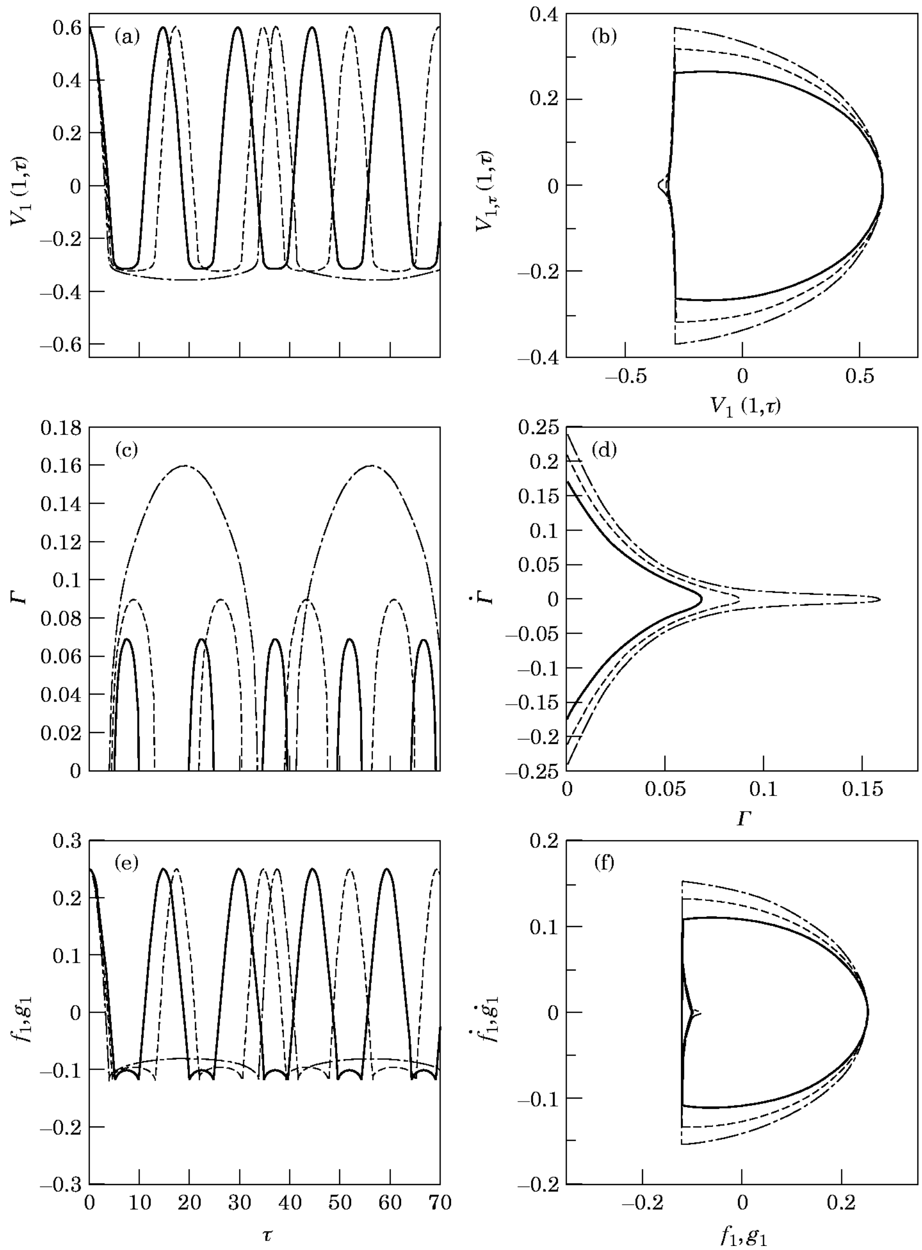

Figure 9. Transient amplitudes and phase planes for various uniform loads. (a) Transient amplitude at $\xi = 1$; (b) phase plane of the amplitude at $\xi = 1$; (c) transient contact length; (d) phase plane of the contact length; (e) transient amplitude; (f) phase plane of the amplitude. —, Q_0 values: 0·01; – –, 0·02; – · –, 0·03; the other parameters are as Figure 6a.

where $\varepsilon = \frac{1}{2}(\Omega_N - \Omega_R)$ is a small positive quantity. The function $\sin \varepsilon\tau$ varies slowly with a large period $2\pi/\varepsilon$. Thus, equation (68) may be regarded as the beating vibration with period $2\pi/\varepsilon$ and variable amplitude $-Q_1 K_{41}/\Omega_N^2 - \Omega_R^2 \sin \varepsilon\tau$. The beating phenomena are shown in Figures 10(a), (c) and (e).

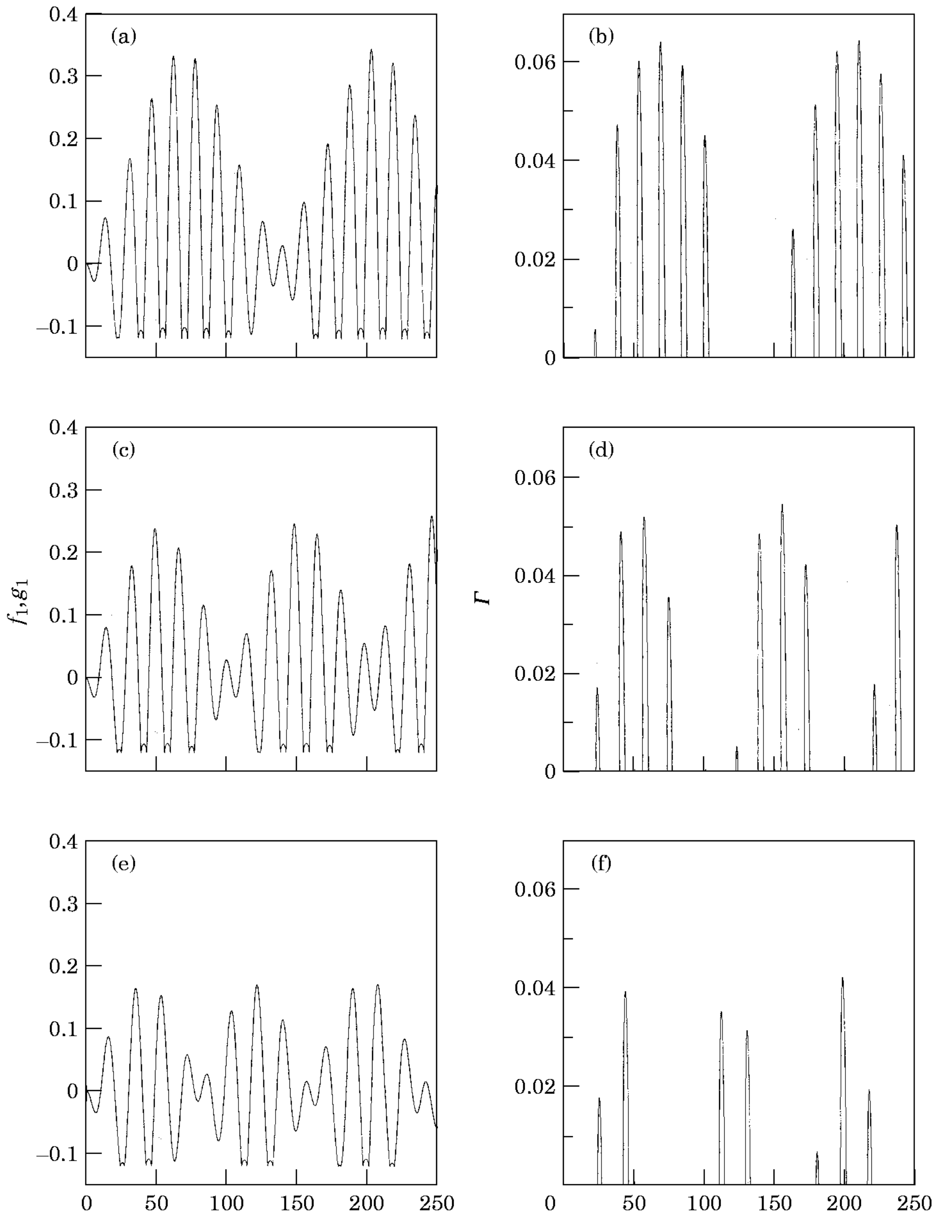

Figure 10. Transient amplitude and transient contact length for various harmonic excitation frequencies. Values of Q_1: (10a, b), 0·005 cos λ_1^2/π^2; (10c, d), 0·005 cos 0·9λ_1^2/π^2; (10e, f), 0·005 cos 0·8λ_1^2/π^2; the other parameters are the same as Figure 6a.

6. CONCLUSIONS

Calculus of variation and Hamilton's principle are used to formulate governing equations, transversality equation and boundary conditions of a beam in contact with a cylindrical foundation. A special numerical technique is developed to solve such a moving boundary problem. The results are simulated by the Runge–Kutta algorithm.

From the numerical results of free vibration, the following conclusions are drawn:

(1) The period of a vibrating beam with constraint is shorter than that without constraint.

(2) Increasing the initial condition will cause an increase in the contact length. As the initial condition decreases, the contact time will decrease.

(3) As the radius parameter of the cylindrical foundation increases, the contact length increases and the period of the vibrating beam decreases.

From numerical results of forced vibration, the following conclusions are drawn:

(1) The period of a vibrating beam increases as the static load increases.

(2) The maximum value of transient amplitude increases when the distributed load increases.

(3) As the static load increases, the contact length increases but the relationship is not linear.

(4) The vibrating beam which is in contact with the cylindrical foundation exhibits beating under resonant excitation.

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APPENDIX A

The details of $J_{1nm}(\Gamma)$, $J_{2nm}(\Gamma)$, $J_{3m}(\Gamma)$ and $J_{4m}(\Gamma)$ are shown in the following

$$J_{1nm}(\Gamma) = \sqrt{\lambda_m \lambda_n / \Omega_m \Omega_n}\{[0{\cdot}5K_{1nm} + \lambda_n(K_{2nm} - K_{6nm})]/[(1-\Gamma)], \tag{A1}$$

$$J_{2nm}(\Gamma) = \sqrt{\lambda_m \lambda_n / \Omega_m \Omega_n}\left\{\frac{0{\cdot}75K_{1nm} + 3\lambda_n(K_{2nm} - K_{6nm}) + \lambda_n^2(K_{3nm} + K_{7nm} - 2K_{8nm})}{(1-\Gamma)^2}\right\} \tag{A2}$$

$$J_{3m}(\Gamma) = \sqrt{\lambda_m/\Omega_m}\{\alpha^2\sqrt{1-\Gamma}K_{4m}/[(\alpha^2-\Gamma^2)^{3/2}] - 3\alpha^2\Gamma(1-\Gamma)^{3/2}K_{5m}/(\alpha^2-\Gamma^2)^{5/2}\} \tag{A3}$$

$$J_{4m}(\Gamma) = -\sqrt{\lambda_m/\Omega_m}\{\alpha^2(1-\Gamma)^{3/2}K_{5m}/(\alpha^2-\Gamma^2)^{3/2}\} \tag{A4}$$

In the above equations, the Γ value is a function of the non-dimensional time τ. The detailed functions $J_{1nm}(\Gamma)$, $J_{2nm}(\Gamma)$, $J_{3m}(\Gamma)$, and $J_{4m}(\Gamma)$, and the detailed constants K_{1nm}, K_{2nm}, K_{3nm}, K_{4m}, K_{5m}, K_{6nm}, K_{7nm}, and K_{8nm} are described in Chen [22].

APPENDIX B: NOMENCLATURE

a	radius of the rigid circular foundation.
A	cross-sectional area of the cantilever beam.
E	Young's modulus.
f_n	nth transient amplitude in the upward configuration.
g_n	nth transient amplitude in the downward configuration.
$\mathbf{i}$, $\mathbf{j}$	unit vectors in the s and y directions, respectively.
I	moment of inertia of cross-section.
l	length.
q, q_0, q_1	external excitations.
Q, Q_0, Q_1	non-dimensional external excitations.
$\mathbf{R}^i$	position vector before deformation.
$\mathbf{R}^f$	position vector after deformation.
$\mathbf{R}_r$	radial vector.
t	time.
$\mathbf{U}$	displacement field.
v	transverse displacement.
V_1, V_2	non-dimensional transverse deflection in the upward and downward configurations, respectively.
α	non-dimensional radius parameter of the rigid circular foundation.

β_n	nth natural frequency in downward configuration.
β'_n	nth natural frequency in downward configuration.
γ	contact length.
Γ	non-dimensional contact length.
ρ	mass per unit length.
ε_{ij}	strains.
κ	curvature.
λ	Langrangian multiplier.
λ_n	nth mode natural frequency.
Λ_n	constant related to frequency λ_n.
ξ	non-dimensional location in s direction.
σ_{in}	stresses.
τ	non-dimensional time.
τ_t	transferring time.
ϕ_n, ψ_n	nth mode shape in the upward and downward configurations, respectively.
ω_f, ω_T	non-dimensional frequencies.
Ω_N	non-dimensional natural frequency.
Ω_R	non-dimensional frequency ratio.
Ω_n	contact related to frequency λ_n.